Some Notes on the Sample Complexity of Approximate Channel Simulation

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Paper

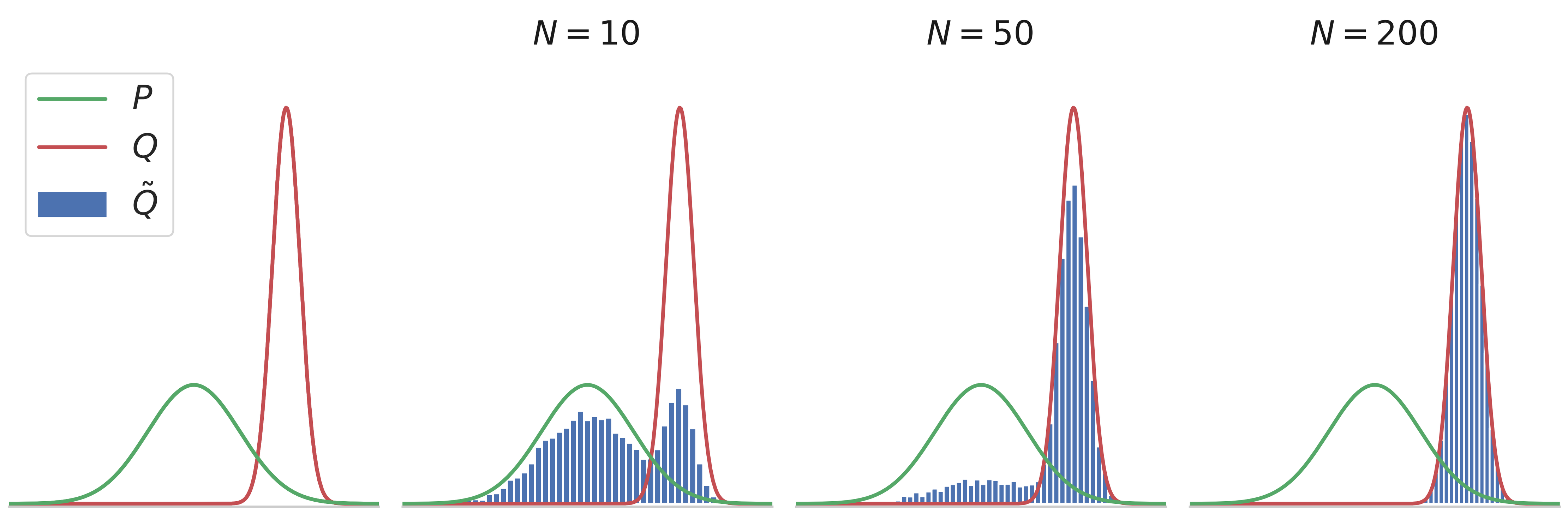
TL;DR: We prove complexity bounds for approximate sampling, improving the results of Block and Polyanskiy (2023).

Problem Setup and Motivation

Have: target Q, samples $X_1, X_2, \ldots, X_N \sim P^{\otimes N}$

Want: $Y \sim \widetilde{Q}$

How do we pick N so that $D_{TV}[Q \parallel \widetilde{Q}] \leq \epsilon$?



Rejection Sampling

Usual selection rule: $K = \min\left\{k \in \mathbb{N} \mid U_k \leq \frac{dQ}{dP}(X_k) \Big/ \left\|\frac{dQ}{dP}\right\|_\infty\right\}$

"Impatient" selection rule: $K_N = \begin{cases} K & \text{if } K \leq N \\ 1 & \text{otherwise.} \end{cases}$

Bound TV:

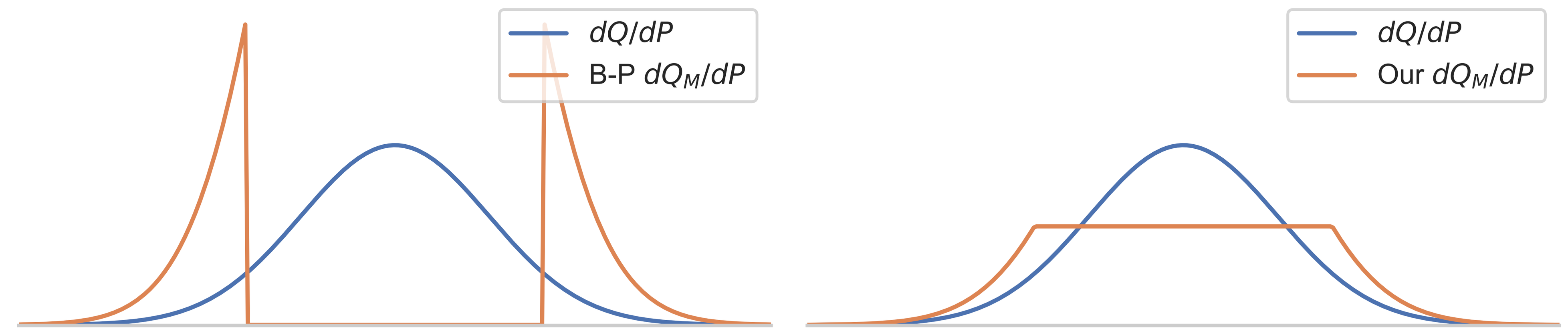
$$D_{TV}[Q \parallel \widetilde{Q}] = \mathbb{P}[K > N] \cdot D_{TV}[Q \parallel P] \leq \mathbb{P}[K > N] \leq \exp\left(-N \Big/ \left\|\frac{dQ}{dP}\right\|_\infty\right)$$

Control $\|dQ/dP\|_\infty$: "chop off" top of dQ/dP.

Approximate Rejection Sampling Complexity (Block and Polyanskiy, 2023)

$$N \geq \frac{2}{1-\epsilon} \log\left(\frac{2}{\epsilon}\right) (f')^{-1}\left(\frac{4 \cdot D_f[Q \parallel P]}{\epsilon}\right) \quad \Longrightarrow \quad D_{TV}[Q \parallel \widetilde{Q}] \leq \epsilon$$

Improved Approximate Rejection Sampling



Improved Approximate Rejection Sampling Complexity

$$\forall \gamma \in (0,1): \quad N \geq \log\left(\frac{1}{(1-\gamma)\epsilon}\right) (f')^{-1}\left(\frac{D_f[Q \parallel P]}{\gamma\epsilon}\right) \quad \Longrightarrow \quad D_{TV}[Q \parallel \widetilde{Q}] \leq \epsilon$$

Approximate Poisson Functional Representation

Poisson process $T_1, T_2, \ldots$

PFR selection rule: $K = \arg\min_{k \in \mathbb{N}} \left\{T_k \Big/ \frac{dQ}{dP}(X_k)\right\}$

Markov:

$$\mathbb{P}[K > N] = \mathbb{P}[\log K > \log N] \leq \frac{\mathbb{E}[\log K]}{\log N}$$

Li and El Gamal (2018): $\mathbb{E}[\log K] \leq D_{\mathrm{KL}}[Q \parallel P] + e^{-1} + \log 2$

PFR sample complexity

$$N \geq \exp\left(\frac{D_{\mathrm{KL}}[Q \parallel P] + e^{-1} + \log 2}{\epsilon}\right) \quad \Longrightarrow \quad D_{TV}[Q \parallel \widetilde{Q}] \leq \epsilon$$

References

Block, A. and Polyanskiy, Y. (2023). The sample complexity of approximate rejection sampling with applications to smoothed online learning. In *COLT 2023*, pages 228–273. PMLR.

Li, C. T. and El Gamal, A. (2018). Strong functional representation lemma and applications to coding theorems. *IEEE Transactions on Information Theory*, 64(11):6967–6978.